Exact SOHS decompositions of trigonometric univariate polynomials with Gaussian coefficients

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ABSTRACT

Certifying the positivity of trigonometric polynomials is of first importance for design problems in discrete-time signal processing. It is well known from the Riesz-Fejéz spectral factorization theorem that any trigonometric univariate polynomial positive on the unit circle can be decomposed as a Hermitian square with complex coefficients. Here we focus on the case of polynomials with Gaussian integer coefficients, i.e., with real and imaginary parts being integers.

We design, analyze and compare, theoretically and practically, three hybrid numeric-symbolic algorithms computing weighted sums of Hermitian squares decompositions for trigonometric univariate polynomials positive on the unit circle with Gaussian coefficients. The numerical steps the first and second algorithm rely on are complex root isolation and semidefinite programming, respectively. An exact sum of Hermitian squares decomposition is obtained thanks to compensation techniques. The third algorithm, also based on complex semidefinite programming, is an adaptation of the rounding and projection algorithm by Peyrl and Parrilo.

For all three algorithms, we prove bit complexity and output size estimates that are polynomial in the degree of the input and linear in the maximum bitsize of its coefficients.

We compare their performance on randomly chosen benchmarks, and further design a certified finite impulse filter.

KEYWORDS

Hybrid symbolic-numeric algorithm, positive trigonometric polynomials, Gaussian coefficients, semidefinite programming, root isolation, filter design

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1 INTRODUCTION

In this paper, we denote by (resp. $\mathbb{N}$) $\mathbb{Z}$ the set of (resp. non-negative) integers and by $\mathbb{Q}$ (resp. $\mathbb{Q}_+$), $\mathbb{R}$ and $\mathbb{C}$ the fields of rational (resp. positive rational), real and complex numbers. For a complex variable or number f, we denote by $\bar{f}$ the associated conjugate variable or number.

Let $\mathscr{H}[z]$ be the set of trigonometric univariate polynomials defined as a subset of Laurent polynomials with complex coefficients and complex variable z as follows:

$$f(z) = f_0 + \left(\frac{f_1}{z} + \bar{f}_1 z\right) + \cdots + \left(\frac{f_d}{z^d} + \bar{f}_d z^d\right), \tag{1}$$

with $f_0 \in \mathbb{R}$ and $d \in \mathbb{N}$. By convention, when $f_d \neq 0$, d is the degree of f; the degree of the zero polynomial is $-\infty$.

In this paper, since we work with base fields of characteristic zero, we see more polynomials through the evaluation maps they define than as algebraic objects. Note that for $f \in \mathscr{H}[z]$, the restriction of the map $\zeta \mapsto f(\zeta)$ over the *unit circle* $\mathscr{C} := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ coincides with the evaluation map defined by the polynomial

$$g(z) = f_0 + \left(f_1 \bar{z} + \bar{f}_1 z\right) + \cdots + \left(f_d \bar{z}^d + \bar{f}_d z^d\right),$$

since $\bar{\zeta} = \zeta^{-1}$ for $\zeta \in \mathscr{C}$. Note also that for any $\zeta \in \mathscr{C}$, $g(\zeta) \in \mathbb{R}$ so that g is a *Hermitian* polynomial. Finally, note that for any Hermitian polynomial g, there exists $f \in \mathscr{H}[z]$ such that the restrictions to $\mathscr{C}$ of the maps $\zeta \mapsto g(\zeta)$ and $\zeta \mapsto f(\zeta)$ coincide (this is due to the fact that all points in $\mathscr{C}$ satsify $\zeta\bar{\zeta} = 1$).

For $h(z) = \sum_{k=0}^{d} h_k z^k \in \mathbb{C}[z]$, we define $h^\star(z) = \sum_{k=0}^{d} \bar{h}_k z^{-k}$.

One says that f is a *sum of Hermitian squares* (SOHS) if there exist some $r \in \mathbb{N} - \{0\}$ and polynomials $s_1, \ldots, s_r$ in $\mathbb{C}[z]$ such that

$$f(z) = \sum_{j=1}^{r} s_j(z) s_j^\star(z)\,.$$

This terminology of Hermitian squares comes from the above discussion as $s_j^\star(\zeta) = s_j(\bar{\zeta})$ for all $\zeta \in \mathscr{C}$. By the Riesz-Fejéz spectral factorization theorem (see, e.g., [9, Theorem 1.1]), any trigonometric univariate polynomial which is non-negative over the unit circle $\mathscr{C}$ can be written as an Hermitian square, i.e., an SOHS with a single term.

Its proof [9, pp. 3–5] is constructive but requires to manipulate exactly all $2d$ complex roots of the associated Laurent polynomial. Usually, this algorithm is applied with approximate computations, leading to approximate certificates of non-negativity over $\mathscr{C}$.

The subset of $\mathscr{H}[z]$ with *Gaussian integers* coefficients, i.e., all coefficients f_i lie in $\mathbb{Z}+i\mathbb{Z}$ (where i is the standard imaginary unit) is denoted by $\mathscr{H}(\mathbb{Z})[z]$. In this paper, we focus on the computation of *exact* certificates of non-negativity of polynomials in $\mathscr{H}(\mathbb{Z})[z]$ by means of *exact* SOHS decompositions.

Our motivation comes from design problems in discrete-time signal processing. In particular, for the design of finite impulse response (FIR) filters in signal processing, minimizing the stopband energy is a crucial issue [9, Chapter 5]. Computing exact SOHS decompositions of trigonometric univariate polynomials in this context appears to be a natural computational issue.

Our contribution. We design three exact algorithms to compute SOHS decompositions of polynomials in $\mathscr{H}(\mathbb{Z})[z]$ that are *positive* over the unit circle $\mathscr{C}$. These algorithms are based on perturbation-compensation or rounding-projection techniques. We analyze their bit complexities and output size as well.

We use the height of a polynomial with rational coefficients to measure its *bitsize* that is defined as follows. The bitsize of an integer b is denoted by $\mathrm{ht}(b) := \lfloor \log_2(|b|) \rfloor + 1$ with $\mathrm{ht}(0) := 1$, where $\log_2$ is the binary logarithm. Given $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ with $b \neq 0$ and $\gcd(a,b) = 1$, we define $\mathrm{ht}\,(a/b) = \mathrm{ht}(a) + \mathrm{ht}(b)$. We define the bitsize of a Gaussian rational number as $\mathrm{ht}(a+ib) = \max(\mathrm{ht}(a), \mathrm{ht}(b))$, where $a, b \in \mathbb{Q}$. For a non-zero polynomial f with Gaussian rational coefficients, we define $\mathrm{ht}(f)$ as the maximum bitsize of the non-zero coefficients of f.

For two maps $p, q : \mathbb{N}^m \to \mathbb{R}$, one writes "$p(v) = O(q(v))$" when there exists $b \in \mathbb{N}$ such that $p(v) \leq bq(v)$, for all $v \in \mathbb{N}^m$. We use the notation $p(v) = \widetilde{O}(q(v))$ when $p(v) = O(q(v)\log^k q(v))$ for some $k \in \mathbb{N}$.

The first algorithm we design, called `csos1`, is a perturbation-compensation one in which the numerical step computes an approximate SOHS decomposition for a well-chosen perturbation of the input polynomial with complex root isolation.

Theorem 1. *Let $f \in \mathscr{H}(\mathbb{Z})[z]$ be positive on $\mathscr{C}$ of degree d and coefficients of maximum bitsize τ. There exists an Algorithm* `csos1` *which on input f computes an SOHS decomposition of f with Gaussian (or Gaussian modulus) coefficients using at most*

$$\widetilde{O}\left(d^6(d+\tau)\right)$$

bit operations. In addition, the maximum bitsize of the output coefficients is bounded from above by $\widetilde{O}(d^5(d+\tau))$.

The two other algorithms, called `csos2` and `csos3`, are based on complex *semidefinite programming* (SDP). SDP consists of minimizing a linear function over a set of matrices constrained to have non-negative eigenvalues; see [36]. In `csos2`, we compute an approximate SOHS decomposition for the perturbation by using complex SDP solving. Algorithm `csos3` is an adaptation of the rounding-projection algorithm raised by Peyrl and Parrilo [29]. These algorithms are more expensive compared to the first one because we replace complex root isolation by complex SDP solving. Despite their worse complexity, they allow one to handle constrained optimization problems and to design filters.

Theorem 2. *Let $f \in \mathscr{H}(\mathbb{Z})[z]$ be positive on $\mathscr{C}$ of degree d and coefficients of maximum bitsize τ. There exist algorithms* `csos2` *and or* `csos3` *which on input f output an SOHS decomposition of f with (modulus) of Gaussian coefficients using at most*

$$\widetilde{O}(d^{13}(d+\tau)^2)$$

bit operations. In addition, the maximal bitsize of the output coefficients is bounded from above by $\widetilde{O}(d^6(\tau+d))$.

These algorithms have been implemented using the Julia programming language [5]. We report on practical experiments that Algorithm `csos1` runs faster than the other algorithms; that coincides with the obtained complexity. Furthermore, we rely on `csos3` to design filters in a certified way.

Related works. Computation of exact weighted sums of squares decompositions of a univariate polynomial $f \in \mathbb{Q}[z]$ has been studied in [8, 23, 33]. Many of the techniques developed in this paper are borrowed from these previous works. We also mention [16] which allows to compute certificates of non-negativity of univariate polynomials sharing common real roots with another univariate polynomials. Note that computing sums of squares decompositions of non-negative univariate polynomials with rational coefficients is easier from a complexity viewpoint, according to the estimate in [23, Theorem 4.4].

It should be noted that our SOHS decomposition problems can be translated into sums of squares decompositions of bivariate polynomials with real variables, which are positive on $\mathscr{C}$. This is a topical computational issue popularized by the original papers of Lasserre and Parrilo [19, 28] allowing to compute approximate sums of squares decompositions based on semi-definite programming. Hybrid symbolic-numeric turning these approximate certificates to exact ones are given in [13, 15, 21, 22, 29]. Exact certificates for several special families of polynomials have been given, for instance SAGE/SONC polynomials [24, 35], polynomials lying in the interior of the SOS cone [22], and polynomials whose gradient ideals are zero-dimensional and radical [21].

Applying those results to the bivariate setting we can reduce our problems to, [22, Theorem 16] yields bit complexity and output bitsize estimates which are exponential in the input degree of f and in the maximum bitsize of its coefficients.

Note that in our case, the degree of the SOHS decomposition is known in advance and we estimate the bit complexity of the coefficients involved in this decomposition. In the multivariate case, the situation is more delicate and providing degree bounds is already a challenge. Recent efforts have been pursued for polynomials positive over the *unit sphere* [11, 31], or for more general closed sets of constraints defined by finitely many polynomials [3, 26]. Theorem 2 in [11] provides us a degree bound that is linear in the number of variables while Theorem 1.7 in [3] and Corollary 1 in [26] give bounds that are polynomial in the input degrees and exponential in the number of variables.

2 AUXILIARY RESULTS

For a polynomial $f = f_0 + \cdots + f_d z^d \in \mathbb{C}[z]$ of degree d, the *minimal distance* between the roots $\alpha_1, \ldots, \alpha_d$ of f is defined by

$$\mathrm{sep}(f) := \min\{|\alpha_i - \alpha_j|,\ \alpha_i \neq \alpha_j\}.$$

The *norm* of f is defined as $\|f\| := \sqrt{|f_d|^2 + \cdots + |f_0|^2}$. The following lemma is an immediate consequence of the discussion following the corollary of Theorem 2 from [25].

Lemma 3. *Let $f \in \mathbb{Z}[i][z]$ of degree d and τ be the maximum bitsize of its coefficients. The minimal distance between the roots of f satisfies*

$$\operatorname{sep}(f) \geq \frac{\sqrt{3}}{d^{\frac{d}{2}+1}\|f\|^{d-1}}. \tag{2}$$

Therefore, one needs an accuracy of $\delta = \widetilde{O}(\tau d)$ to compute distinct approximations of the roots of f with complex root isolation.

Proof. By [25, Theorem 2], the minimal distance between the roots of f satisfies:

$$\operatorname{sep}(f) \geq \frac{\sqrt{3|\operatorname{Disc}(f)|}}{d^{\frac{d}{2}+1}\|f\|^{d-1}}, \tag{3}$$

where $\operatorname{Disc}(f) = f_d^{2d-2}\prod_{j<k}(\alpha_j - \alpha_k)^2$ is the discriminant of f. Note that $\operatorname{Disc}(f)$ can be written as a polynomial in $f_0, \ldots, f_d$ with integer coefficients, thus $\operatorname{Disc}(f) \in \mathbb{Z}[i]$. Now we consider the two following cases.

Case 1: f has no multiple root in $\mathbb{C}$. Then, $\operatorname{Disc}(f)$ is nonzero. Since $\operatorname{Disc}(f) \in \mathbb{Z}[i]$, one has $|\operatorname{Disc}(f)| \geq 1$ which from (3), implies

$$\operatorname{sep}(f) \geq \frac{\sqrt{3|\operatorname{Disc}(f)|}}{d^{\frac{d}{2}+1}\|f\|^{d-1}} \geq \frac{\sqrt{3}}{d^{\frac{d}{2}+1}\|f\|^{d-1}}.$$

Case 2: f has a multiple root in $\mathbb{C}$. Let p be the square-free part of f. From [18, Corollary 2.2], the coefficients of p lie in $\mathbb{Z}[i]$. Clearly, $\operatorname{sep}(p) = \operatorname{sep}(f)$ and $\|p\| \leq \|f\|$. Hence, by applying Case 1 for p, where $k = \deg p \leq d$, we have

$$\operatorname{sep}(f) = \operatorname{sep}(p) \geq \frac{\sqrt{3}}{k^{\frac{k}{2}+1}\|p\|^{k-1}} \geq \frac{\sqrt{3}}{d^{\frac{d}{2}+1}\|f\|^{d-1}}.$$

□

The following lemma provides a lower bound on the minimum of a real bivariate polynomial over the unit circle in $\mathbb{R}^2$.

Lemma 4. *Let $p \in \mathbb{Z}[x, y]$ be a real bivariate polynomial of degree d and τ be the maximum bitsize of its coefficients. Assume that p is positive on the unit circle $\mathscr{C}$. Then, the minimum of p on $\mathscr{C}$ satisfies the following inequality:*

$$p_{\min} := \min\{p(x, y) : x^2 + y^2 = 1\} \geq 2^{-\widetilde{O}(d^3(d+\tau))}. \tag{4}$$

Proof. We consider the following algebraic set:

$$V := \{(x, y, m) \in \mathbb{C}^3 : p(x, y) - m = y\frac{\partial p}{\partial x} - x\frac{\partial p}{\partial y} = 0, x^2 + y^2 = 1\}.$$

Note that the projection of V on the m-axis defines the critical values of the restriction of the evaluation map $z \mapsto p(z)$ to $\mathscr{C}$ which contains $p_{\min}$.

Assume first that V is finite. By [32, Corollary 2], there is a zero-dimensional parametrization of V defined by real univariate polynomials with bitsizes upper bounded by $\widetilde{O}(d^3(d+\tau))$. Since there exists (x_0, y_0) on $\mathscr{C}$ such that $(x_0, y_0, p_{\min})$ belongs to V, $p_{\min}$ is a (non-zero) root of a univariate polynomial of degree at most $O(d^3\tau)$. Hence, the Cauchy bound [7] yields:

$$|p_{\min}| \geq 2^{-\widetilde{O}(d^3(d+\tau))}.$$

Assume now that V is not finite. By Krull's theorem [17], this implies that $\mathscr{C}$ is contained in the complex zero set defined by $y\frac{\partial p}{\partial x} - x\frac{\partial p}{\partial y} = 0$, whence is a factor of this polynomial. This implies that there exists a factorization $p = p_1 p_2$ where p_1 is a power of x^2+y^2-c (where c is a constant) and the zero set of the polynomial $y\frac{\partial p_2}{\partial x} - x\frac{\partial p_2}{\partial y} = 0$ has a zero-dimensional intersection with $\mathscr{C}$. This yields the following analysis. The set V is the union of a 1-dimensional component containing points (m, x, y) where (x, y) ranges over $\mathscr{C}$ and $m = c - 1$, and a 0-dimensional component containing points (m, x, y) which are solutions to

$$p(x, y) = m,\ y\frac{\partial p_2}{\partial x} - x\frac{\partial p_2}{\partial y} = 0,\ x^2 + y^2 = 1.$$

Applying the first paragraph of the proof to the above system ends the proof. □

The upcoming result will be used to estimate bit complexities of the two algorithms `csos1` and `csos2`.

Lemma 5. *Let $f \in \mathscr{H}(\mathbb{Z})[z]$ be positive on $\mathscr{C}$, of degree d and τ be the maximum bitsize of its coefficients. Then, there exists a positive integer $N = \widetilde{O}(d^3(d+\tau))$ such that $f - \frac{1}{2^N}$ is positive on $\mathscr{C}$.*

Proof. With $z = x + iy$, let us define $p(x, y) := f(x + iy)$. Since $f \in \mathscr{H}(\mathbb{Z})[z]$, one has $p \in \mathbb{Z}[x, y]$ with degree $2d$ and bitsize $O(\mathrm{ht}(d) + \tau)$. Clearly, $\min\{f(z) : |z| = 1\} = \min\{p(x, y) : x^2 + y^2 = 1\} = p_{\min}$. Let us choose a positive integer N such that $\frac{1}{2^N} \leq p_{\min}$. From Lemma 4, we conclude that $N = \widetilde{O}(d^3(d + \tau))$. □

The following result, stated in [2, Lemma 2.1 & Theorem 3.2], will be used to investigate the bit complexity of the two algorithms `csos2` and `csos3` based on SDP solving.

Lemma 6. *Let Q be a Hermitian matrix indexed on $\{-d, \ldots, d\}$, with positive eigenvalues and rational entries. Let L be the factor of Q computed by Cholesky's decomposition with finite precision δ_c. Then, $LL^T = Q + H$, where*

$$|H_{ij}| \leq \frac{(d+2)2^{-\delta_c}\sqrt{|Q_{ii}Q_{jj}|}}{1 - (d+2)2^{-\delta_c}}. \tag{5}$$

In addition, if the smallest eigenvalue $\tilde{\lambda}$ of Q satisfies the inequality

$$2^{-\delta_c} < \frac{\tilde{\lambda}}{d^2 + d + (d-1)\tilde{\lambda}}, \tag{6}$$

Cholesky's decomposition returns a rational nonsingular factor L.

3 ALGORITHM BASED ON ROOT ISOLATION

In this section, we propose an algorithm, called `csos1`, to compute an SOHS decomposition of a polynomial in $\mathscr{H}(\mathbb{Z})[z]$ which is positive on the unit circle $\mathscr{C}$. It puts into practice a perturbation-compensation procedure based on complex roots isolation, and can be viewed as the extension of the procedure `univsos2` (stated in [8] and analyzed in [23, § 4]) to the complex setting.

3.1 Description

Description. Algorithm csos1 takes as input a polynomial $f \in \mathscr{H}(\mathbb{Z})[z]$ of degree d which is positive on $\mathscr{C}$. It outputs two positive rational numbers ε, a, a rational number u_0, and two lists of Gaussian numbers $[u_1, \ldots, u_d]$ and $[\alpha_1, \ldots, \alpha_d]$ such that

$$f(z) = \left(\varepsilon - 2\sum_{k=1}^{d} |u_k| - u_0\right) + \sum_{k=1}^{d} |u_k| \left(z^k + \frac{u_k}{|u_k|}\right)\left(z^{-k} + \frac{\bar{u}_k}{|u_k|}\right) + a \prod_{k=1}^{d} (z - \alpha_k)\left(z^{-k} - \bar{\alpha}_k\right) \text{ with } \left(\varepsilon - 2\sum_{k=1}^{d} |u_k| - u_0\right) > 0. \quad (7)$$

Algorithm 1 csos1

Input: $f \in \mathscr{H}(\mathbb{Z})[z]$ positive on $\mathscr{C}$ of degree d

Output: $\varepsilon, a \in \mathbb{Q}_+$, $u_0 \in \mathbb{Q}$, two lists $[u_1, \ldots, u_d]$ and $[\alpha_1, \ldots, \alpha_d]$ in $\mathbb{Q}[i]$ providing an SOHS decomposition for f on $\mathscr{C}$ as in (7):

1: $\delta := 1$, $\varepsilon := 1$ and $p := f(x + iy)$ ▷ $z = x + iy$
2: **while** hasrealrootoncircle$(p - \varepsilon)$ **do** $\varepsilon := \frac{\varepsilon}{2}$
3: **done**
4: boo := false
5: **while** not boo **do**
6: $[\alpha_1, \ldots, \alpha_d]$:= complexroots$(f - \varepsilon, \delta)$
7: $F := \prod_{k=1}^{d} (z - \alpha_k)\left(z^{-1} - \bar{\alpha}_k\right)$
8: $a :=$ coeffs$(f, 0)$/coeffs$(F, 0)$, $u := f - \varepsilon - aF$
9: $[u_0, u_1, \ldots, u_d]$:= coeffs(u)
10: **if** $\varepsilon > u_0 + 2\sum_{k=1}^{d} |u_k|$ **then** boo := true
11: **else** $\delta := 2\delta$
12: **end**
13: **done**
14: **return** $\varepsilon, a, u_0, [u_1, \ldots, u_d], [\alpha_1, \ldots, \alpha_d]$

In Line 1 we replace z by $x+iy$ in f where x, y are (real) variables to obtain a real bivariate polynomial p of degree $2d$. Since, by assumption, f is positive over $\mathscr{C}$, there exists $\varepsilon > 0$ small enough, such that p is positive on $\mathscr{C}$. The first while loop from Line 2 to 3 computes such positive rational number ε. To do so, it uses an auxiliary procedure hasrealrootoncircle, which returns true if $p - \varepsilon$ cancels on the $\mathscr{C} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 1 = 0\}$. Such a procedure is easily obtained with any polynomial system solver for bivariate polynomial systems. In practice we use the real root solver msolve [4].

In the second while loop from Line 5 to 13, the algorithm computes at Line 6 Gaussian approximations $\alpha_1, \ldots, \alpha_d$ (and their conjugates) of the complex roots of $f - \varepsilon$ with accuracy δ. This is done using a procedure complexroots which on input a rational fraction and a required accuracy δ returns all the complex roots of the numerator of the fraction at accuracy δ (see, e.g., [6]).

The idea is to obtain (up to proper scaling) an approximate SOHS decomposition F of $f - \varepsilon$.

The auxiliary coeffs procedure provides the list of coefficients of a polynomial, e.g., coeffs$(f, 0)$ returns the constant term of f. We then consider the remainder u at Line 8 which is the difference between $f - \varepsilon$ and its approximate SOHS decomposition. As proved in Section 3.2, if the precision of root isolation is large enough, the stopping condition $\varepsilon > u_0 + 2\sum_{k=1}^{d} |u_k|$ is fulfilled, otherwise the precision is increased.

To illustrate csos1, we use the following simple example.

Example 7. Let $f = 5 + (1 + i)z^{-1} + (1 - i)z$ which is positive on $\mathscr{C}$. We obtain $p = 5 + 2x + 2y$. With $\varepsilon = 1$, we check with hasrealrootoncircle that $p - \varepsilon$ is positive on $\mathscr{C}$. With precision $\delta = 16$, we compute complex approximation roots $\alpha_1 = -\frac{7}{4} - \frac{7}{4}i$ and $\bar{\alpha}_1$ of $f - \varepsilon$. Defining $F = (z - \alpha_1)(z^{-1} - \bar{\alpha}_1)$, we obtain $a = \frac{32}{57}$, $u = f - \varepsilon - aF = (\frac{1}{57} + \frac{i}{57})z^{-1} + (\frac{1}{57} - \frac{i}{57})z$. Clearly, $\varepsilon = 1 > 0 + \frac{2\sqrt{2}}{57}$ so the condition in Line 10 is satisfied. Then, f has an exact SOHS decomposition as follows:

$$f = (1 - \tfrac{2\sqrt{2}}{57}) + \tfrac{\sqrt{2}}{57}(z + \tfrac{1+i}{\sqrt{2}})(z^{-1} + \tfrac{1-i}{\sqrt{2}}) + \tfrac{32}{57}(z + \tfrac{7}{4} + \tfrac{7}{4}i)(z^{-1} + \tfrac{7}{4} - \tfrac{7}{4}i).$$

3.2 Proof of Theorem 1

Correctness of Algorithm csos1. We first prove that *Algorithm* csos1 *terminates and outputs an SOSH decomposition of* f.

By Lemma 5, there exits a positive rational ε such that $f - \varepsilon$ is also positive on $\mathscr{C}$. Thus, the first loop (from Line 2 to Line 3) of Algorithm csos1 terminates. The magnitude of the coefficients of the remainder polynomial u defined in Line 9 converges to 0 as the precision δ of the complex root finder goes to infinity (because of the continuity of roots w.r.t. coefficients). This implies that the condition of Line 10 is fulfilled after finitely many iterations, thus the second loop (from Line 5 to Line 13) always terminates. Eventually, we have

$$f = \varepsilon + u_0 + \left(u_1 z^{-1} + \bar{u}_1 z\right) + \cdots + \left(u_d z^{-d} + \bar{u}_d z^d\right) + aF.$$

In addition,

$$u_k z^{-k} + \bar{u}_k z^k = |u_k| \left(z^k + \frac{u_k}{|u_k|}\right)\left(z^{-k} + \frac{\bar{u}_k}{|u_k|}\right) - 2|u_k|, \quad (8)$$

yielding (7). From the proof of the Riesz-Fejéz theorem (see, e.g., [9, pp. 3–5]), f can be decomposed as a single Hermitian square, thus the constant term of f lies in $\mathbb{Q}_+$. Similarly the constant term $\prod_{k=1}^{d} |\alpha_k|^2$ of F is also a positive rational number, so $a \in \mathbb{Q}_+$. Clearly, the polynomial F and the first term on the right-hand side of (8) are SOHS. Hence, as $\varepsilon > u_0 + 2\sum_{k=1}^{d} |u_k|$, the right-hand side of (7) is a sum of $d + 2$ Hermitian squares involving Gaussian (or Gaussian modulus) numbers. □

We now analyze the bit complexity of Algorithm csos1.

Proof of Theorem 1. Let us show that *the bitsizes of* $u_0, \ldots, u_d$, $\alpha_1, \ldots, \alpha_d$ *and* a *in* (7) *are bounded from above by* $\widetilde{O}(d^5(d + \tau))$.

The proof is almost the same as in the univariate real setting [23, Theorem 4.3], thus we only provide the main ingredients and skip some technical details. From Lemma 5, there exists a positive integer $N = \widetilde{O}(d^3(d + \tau))$ such that $f - \varepsilon$ is positive on $\mathscr{C}$, with $\varepsilon = \frac{1}{2^N}$. Define $m := 2d$ and $g(z) := z^d(f - \varepsilon) = \bar{f}_d z^{2d} + \cdots + \bar{f}_1 z^{d+1} + g_d z^d + f_1 z^{d-1} + \cdots + f_d$, with $g_d = f_0 - \varepsilon$. Note that g and $f - \varepsilon$ have the same roots. Denote by $\zeta_1, \ldots, \zeta_m$ the (exact) complex roots of g and by $\zeta'_1, \ldots, \zeta'_m$ their approximations with a precision δ, so that $\zeta'_j = \zeta_j(1 + e_j)$, where $|e_j| \le e := 2^{-\delta}$, for $j = 1, \ldots, m$.

Let $g' := \bar{f}_d(z - \zeta'_1) \ldots (z - \zeta'_m)$. The remainder polynomial u defined in Line 8 of Algorithm csos1 satisfies $z^d u = g - g'$.

We now show that if $\delta = N + \log_2((2d+1)^2\|f\|) = \widetilde{O}(d^3(d+\tau))$, then the coefficients of u satisfy the condition $\varepsilon > u_0 + 2\sum_{k=1}^{d}|u_k|$. We apply Lemma 3 to the polynomial obtained by multiplying $z^d(f-\varepsilon)$ with the least common multiple of its coefficients. Hence, we require an accuracy at least $\widetilde{O}(Nd)$ to compute distinct approximations of its roots in the worst case. We will actually need an accuracy of larger bitsize $\widetilde{O}(d^4(d+\tau))$ in the worst case. Let $j \in \{0, 1, \ldots, d\}$. Using Vieta's formulas, we have

$$\sum_{1\le i_1<\cdots<i_j\le m} \zeta_{i_1}\cdots\zeta_{i_j} = (-1)^j\frac{g_{m-j}}{g_m} = (-1)^j\frac{g_{m-j}}{\bar{f}_d}. \quad (9)$$

Similarly, we have:

$$\sum_{1\le i_1<\cdots<i_j\le m} \zeta'_{i_1}\cdots\zeta'_{i_j} = (-1)^j\frac{g'_{m-j}}{\bar{f}_d}. \quad (10)$$

We estimate an upper bound for the coefficient $\bar{u}_{d-j}$ of the difference polynomial u. Clearly, $\bar{u}_{d-j} = g_{m-j} - g'_{m-j}$. From (9) and (10), we see that

$$\begin{aligned}|\bar{u}_{d-j}| &= |\bar{f}_d|\left|\sum_{1\le i_1<\cdots<i_{d+j}\le m}(\zeta_{i_1}\cdots\zeta_{i_{d+j}} - \zeta'_{i_1}\cdots\zeta'_{i_{d+j}})\right| \\ &= |\bar{f}_d|\left|\sum_{1\le i_1<\cdots<i_{d+j}\le m}\zeta_{i_1}\cdots\zeta_{i_{d+j}}\left(1-\prod_{k=1}^{d+j}(1+e_{i_k})\right)\right|.\end{aligned}$$

Then, exactly as in the proof of [23, Theorem 4.3], we rely on [14, Lemma 3.3] and the Cauchy bound [7] to obtain the desired estimates.

We now prove the remaining assertion in Theorem 1: *Algorithm* `csos1` *runs in* $\widetilde{O}(d^6(d+\tau))$ *boolean operations.* Again the proof scheme is very similar as the one of [23, Theorem 4.4]. The algorithm includes two steps.

We consider the first step checking that $g(z)$ defined in the previous proof has no real root on the unit circle. Let ε be given as in Lemma 5 with $\mathrm{ht}(\varepsilon) = \widetilde{O}(d^3(d+\tau))$. By relying on Sylvester–Habicht sequences [20, Corollary 5.2], the check can be performed using $O((2d)^2\,\mathrm{ht}(\varepsilon)) = \widetilde{O}(d^5(d+\tau))$ boolean operations. In the second step, we compute approximate complex roots of $g(z)$ and check the condition at Line 10. It follows from [27, Theorem 4] that isolating disks of radius less than $2^{-\delta}$ for all complex roots of $g(z)$ can be computed in $\widetilde{O}(d^3 + d^2\,\mathrm{ht}(\varepsilon) + d\delta)) = \widetilde{O}(d^6(d+\tau))$ boolean operations. The computation of all u_k has a negligible cost w.r.t. to the computation of the complex roots. Therefore, we conclude that `csos1` runs in $\widetilde{O}(d^6(d+\tau))$ boolean operations. □

4 ALGORITHM BASED ON COMPLEX SDP

This section states and analyzes another perturbation-compensation algorithm, named `csos2`, to compute an SOHS decomposition of a trigonometric polynomial being positive on $\mathscr{C}$. In the algorithm, the approximate SOHS decomposition for the perturbation is computed by using complex SDP solving. It can be viewed as the adaptation of the procedure `intsos` (stated and analyzed in [22, § 3]) to the complex univariate setting. Let I stands for the identity matrix of size $d+1$. A Hermitian matrix Q is said to be positive semidefinite (resp. definite) if Q has only non-negative (resp. positive) eigenvalues, and in this case we use the notation $Q \succeq 0$ (resp. $Q \succ 0$). Given $f \in \mathscr{H}[z]$ of degree d, recall that a Hermitian matrix $Q \in \mathbb{C}^{(d+1)\times(d+1)}$ is called a *Gram* matrix associated with f if $f = v_d^\star \cdot Q \cdot v_d$, where $v_d(z) := (1, z, \ldots, z^d)$ contains the canonical basis for polynomials of degree d in z. By [9, Theorem 2.5], f is positive on $\mathscr{C}$ if and only if there exists a positive definite Gram matrix associated to f.

4.1 Description

The input of Algorithm `csos2` includes a polynomial $f \in \mathscr{H}(\mathbb{Z})[z]$ of degree d which is positive on $\mathscr{C}$. The outputs are $\varepsilon \in \mathbb{Q}_+$, a list of Gaussian numbers $[u_0, u_1, \ldots, u_d]$, and a list of polynomials $[s_1, \ldots, s_d]$ in $\mathbb{Q}[i][z]$ providing an SOHS decomposition of f as follows

$$f = \Big(\varepsilon - u_0 - 2\sum_{k=1}^{d}|u_k|\Big) + \sum_{k=1}^{d}|u_k|\Big(z^k + \frac{u_k}{|u_k|}\Big)\Big(z^{-k} + \frac{\bar{u}_k}{|u_k|}\Big) + \sum_{k=0}^{d} s_k^\star s_k. \quad (11)$$

Algorithm 2 `csos2`

Input: $f \in \mathscr{H}(\mathbb{Z})[z]$ positive on $\mathscr{C}$ of degree d
Output: $\varepsilon \in \mathbb{Q}_+$, $[u_0, u_1, \ldots, u_d]$ in $\mathbb{Q}[i]$, $[s_0, \ldots, s_d]$ in $\mathbb{Q}[i][z]$ providing an SOHS decomposition of f as in (11).

1: $\delta := 1$, $R := 1$, $\delta_c = 1$, $\varepsilon := 1$ and $p := f(x+iy)$
2: **while** `hasrealrootoncircle`$(p-\varepsilon)$ **do** $\varepsilon := \frac{\varepsilon}{2}$
3: **done**
4: boo := false
5: **while** not boo **do**
6: $\quad(\tilde{Q}, \tilde{\lambda}) := \mathsf{sdp}(f-\varepsilon, \delta, R)$
7: $\quad[s_0, \ldots, s_d] := \mathtt{cholesky}(\tilde{Q}, \tilde{\lambda}, \delta_c)$ ▷ $f_\varepsilon \simeq \sum_{k=0}^{d} s_k^\star s_k$
8: $\quad u := f - \varepsilon - \sum_{k=0}^{d} s_k^\star s_k$, $[u_0, u_1, \ldots, u_d] := \mathtt{coeffs}(u)$
9: $\quad$ **if** $\varepsilon > u_0 + 2\sum_{k=1}^{d}|u_k|$ **then** boo := true
10: $\quad$ **else** $\delta := 2\delta$, $R := 2R$, $\delta_c := 2\delta_c$
11: $\quad$ **end**
12: **done**
13: **return** $\varepsilon, [u_0, u_1, \ldots, u_d], [s_0, \ldots, s_d]$

The first steps of `csos2` (Lines 1–3) are exactly the same as `csos1` to obtain $\varepsilon \in \mathbb{Q}_+$ such that $f - \varepsilon$ is positive on $\mathscr{C}$. Then, instead of using root isolation as in `csos1`, `csos2` relies on complex SDP (Line 6) and Cholesky's decomposition (Line 7) to compute an approximate SOHS decomposition of the perturbed polynomial. With $f-\varepsilon$, δ, and R, the `sdp` function calls an SDP solver to compute a rational approximation $\tilde{Q}$ of the Gram matrix associated to $f-\varepsilon$ and a rational approximation $\tilde{\lambda}$ of its smallest eigenvalue. Its outputs are obtained by solving the following complex SDP:

$$\begin{aligned}\lambda_{\min} = \max_{Q,\lambda}\ & \lambda \\ \text{s.t. } & \mathrm{tr}(\Theta_k Q) = f_k - 1_{k=0}\varepsilon, k = -d, \ldots, d, \\ & Q \succeq \lambda I, \lambda \ge 0,\ Q \in \mathbb{C}^{(d+1)\times(d+1)},\end{aligned} \quad (12)$$

where Θ_k is the elementary Toeplitz matrix with ones on the k−th diagonal and zeros elsewhere, $1_{k=0} = 1$ if $k = 0$ and 0 otherwise, $\mathrm{tr}(\cdot)$ stands for the usual matrix trace operator. The equality constraints of SDP (12) corresponds to the relation $f(z) - \varepsilon = v_d^\star(z) \cdot Q \cdot v_d(z)$. This SDP program (corresponding to SDP (2.14) in [9]) computes the Gram matrix associated to f with the largest

minimal eigenvalue. The cholesky function computes first an approximate Cholesky's decomposition LL^T of $\tilde{Q}$ with precision δ_c, and provides as output a list of polynomials $[s_0, \ldots, s_d] \in \mathbb{Q}[i][z]$, s_k is the inner product of the $(k+1)$-th row of L by v_d. One would expect to have $f - \varepsilon = \sum_{k=0}^{d} s_k^\star s_k$ after using exact SDP and Cholesky's decomposition. Since the SDP solver is not exact, we have to consider the remainder $u = f - \varepsilon - \sum_{k=0}^{d} s_k^\star s_k$ and proceed exactly as in csos1 to obtain an exact SOHS decomposition.

4.2 Proof of Theorem 2

The lemma below prepares the bit complexity analysis of csos2.

Lemma 8. *Let $f \in \mathscr{H}(\mathbb{Z})[z]$ be positive on $\mathscr{C}$ of degree d and bitsize τ. Assume that Q is a positive definite Gram matrix associated to f. Then, there exist $\varepsilon \in \mathbb{Q}_+$ of bitsize $\widetilde{O}(d^3(d+\tau))$ such that $f - \varepsilon$ is positive on $\mathscr{C}$, δ of bitsize $\widetilde{O}(d^3(d+\tau))$ and R of bitsize $O(\mathrm{ht}(d) + \tau)$ such that $Q - \varepsilon/(d+1)I$ is a Gram matrix associated to $f - \varepsilon$ with $Q - \varepsilon/(d+1)I \succ 2^{-\delta} I$ and $\sqrt{\mathrm{tr}((Q - \varepsilon I)^2)} \leq R$.*

Proof. By Lemma 5, there is a positive integer N and $\varepsilon = 2^{-N} = \widetilde{O}(d^3(d+\tau))$ such that $f - 3\varepsilon/2 > 0$ on $\mathscr{C}$. Let $\delta := \lceil N + 1 + \log_2(d+1) \rceil = \widetilde{O}(d^3(d+\tau))$ so that $2^{-\delta} \leq \frac{\varepsilon}{2(d+1)}$. One has $v_d^\star(z) v_d(z) = d+1$, thus $f(z) - \varepsilon = v_d^\star(z)(Q - \varepsilon/(d+1)I)v_d(z)$. Since $f(z) - \varepsilon - 2^{-\delta}(d+1) > 0$, we obtain $v_d^\star(z)(Q - \varepsilon/(d+1)I - 2^{-\delta} I)v_d(z) > 0$.

Let $R := \sqrt{(d+1) f_0}$. Note that the equality constraint of SDP (12) with $k = 0$ reads $\mathrm{tr}(Q) = f_0 - \varepsilon \leq f_0$. The maximal eigenvalue of Q is less than f_0 and $\mathrm{tr}((Q - \varepsilon I)^2) \leq \mathrm{tr}(Q^2) \leq (d+1) f_0^2 = R^2$. □

The correctness and bit complexity proofs are very similar to the ones for the intsos algorithm in [22, Proposition 10], so we only provide a sketch with the main ingredients.

Proof of correctness for csos2. Since $f - \varepsilon$ is positive on $\mathscr{C}$, SDP (12) has always a strictly feasible solution for precision parameters (δ, R) with bitsizes as in Lemma 8 and the sdp function returns an approximate Gram matrix $\tilde{Q}$ of $f - \varepsilon$ such that $\tilde{Q} \succeq 2^{-\delta} I$ and $\mathrm{tr}(Q^2) \leq R^2$. In particular, we obtain a rational approximation $\tilde{\lambda} \geq 2^{-\delta}$ of the smallest eigenvalue of $\tilde{Q}$.

At Line 6, we compute an approximate Cholesky decomposition of $\tilde{Q}$ by using the cholesky procedure; we obtain a rational nonsingular factor if there exists δ_c satisfying (6). Let δ_c be the smallest integer such that $2^{-\delta_c} < \frac{2^{-\delta}}{d^2+d+(d-1)2^{-\delta}}$. Since the function $t \mapsto \frac{t}{d^2+d+(d-1)t}$ increases on $[0, +\infty)$ and $\tilde{\lambda} \geq 2^{-\delta}$, (6) holds.

We now consider the polynomial $u = f - \varepsilon - \sum_{k=0}^{d} s_k^\star s_k$. The while loop (Lines 5–12) terminates when $\varepsilon > u_0 + 2\sum_{k=1}^{d} |u_k|$. This condition holds if $|u_k| \leq \frac{\varepsilon}{2d+1}$, for all $k = 0, \ldots, d$. As in the proof of [22, Proposition 10], we prove that this holds for large enough δ and δ_c, with bitsizes $\widetilde{O}(d^3(d+\tau))$. □

Bit complexity estimate for csos2. We prove now that Algorithm csos2 runs in $\widetilde{O}(d^{13}(d+\tau)^2)$ boolean operations.

Assume that ε, δ, R and δ_c are given as above so that, before terminating, Algorithm csos2 performs a single iteration in each while loop. From above results, the bitsize of each $\varepsilon, \delta, \delta_c$ is upper bounded by $\widetilde{O}(d^3(d+\tau))$ and that of R is $O(\mathrm{ht}(d) + \tau)$.

To investigate the computational cost of the call to sdp at Line 6, let us note $n_{\mathrm{sdp}} = d+1$ the size of $\tilde{Q}$ and $m_{\mathrm{sdp}} = 2d+1$ the number of affine constraints of SDP (12). We rely on the bit complexity analysis of the ellipsoid method [30]. Solving SDP (12) is performed in $O(n_{\mathrm{sdp}}^4 \log_2(2^\tau n_{\mathrm{sdp}} R\, 2^\delta))$ iterations of the ellipsoid method, where each iteration requires $O(n_{\mathrm{sdp}}^2(m_{\mathrm{sdp}} + n_{\mathrm{sdp}}))$ arithmetic operations over $\log_2(2^\tau n_{\mathrm{sdp}} R\, 2^\delta)$-bit numbers (see, e.g., [12, 30]). We obtain the following estimates: $O(n_{\mathrm{sdp}}^4 \log_2(2^\tau n_{\mathrm{sdp}} R\, 2^\delta)) = \widetilde{O}(d^7(d+\tau))$, $O(n_{\mathrm{sdp}}^2(m_{\mathrm{sdp}} + n_{\mathrm{sdp}})) = O(d^3)$, and $O(\log_2(2^\tau n_{\mathrm{sdp}} R\, 2^\delta)) = \widetilde{O}(d^3(d+\tau))$. Therefore, the ellipsoid algorithm runs in boolean time $\widetilde{O}(d^{13}(d+\tau)^2)$ to compute the approximate Gram matrix $\tilde{Q}$.

Next, we compute the cost of calling cholesky in Line 7. Note that Cholesky's decomposition is performed in $O(n_{\mathrm{sdp}}^3)$ arithmetic operations over δ_c-bit numbers. Because of $\delta_c = \widetilde{O}(d^3(d+\tau))$ and $n_{\mathrm{sdp}} = d+1$, cholesky runs in boolean time $\widetilde{O}(d^6(d+\tau))$.

The other elementary arithmetic operations of Algorithm csos2 have a negligible cost w.r.t. to the sdp procedure. Hence, the algorithm runs in boolean time $\widetilde{O}(d^{13}(d+\tau)^2)$. □

5 ROUNDING-PROJECTION ALGORITHM

Here, we introduce Algorithm csos3 which is an adaptation of the rounding-projection algorithm by Peyrl and Parrilo, stated in [29] and analyzed in [22, § 3.4], and investigate its bit complexity.

The input of csos3 is a polynomial $f \in \mathscr{H}(\mathbb{Z})[z]$ of degree d which is positive over $\mathscr{C}$. The outputs consist of a list $[c_0, \ldots, c_d] \subset \mathbb{Q}_+$ and a list of polynomials $[s_0, \ldots, s_d]$ in $\mathbb{Q}[i][z]$ that provide an SOHS decomposition of f, namely $f = \sum_{k=0}^{d} c_k s_k^\star s_k$.

As in csos2, the first while loop from Lines 3–8 provides an approximate Gram matrix $\tilde{Q}$ associated to f and an approximation $\tilde{\lambda}$ of its smallest eigenvalue. In Line 11, we round the matrix $\tilde{Q}$ up to precision $\hat{\delta}$ to obtain a matrix $\hat{Q}$, with Gaussian coefficient entries. The for loop from Line 12 to Line 15 is the projection step to ensure that the equality constraints of SDP (12) hold exactly. Then we compute the LDL^T decomposition of Q. The list $[c_0, \ldots, c_d]$ is the list of coefficients of the diagonal matrix D and each s_k is the inner product of the $(k+1)$-th row of L and the vector v_d of all monomials up to degree d. If all c_k's are positive rationals and all polynomials s_k' have Gaussian coefficients, then the second while loop ends, otherwise we increase the precision $\hat{\delta}$.

As emphasized in [22, § 3.4], it turns out that csos2 and csos3 have the same bit complexity. We omit any technicalities as the proof is almost the same as [22, Theorem 12].

Bit complexity estimate for csos3. For $f \in \mathscr{H}(\mathbb{Z})[z]$ positive over $\mathscr{C}$ with degree d and maximal bitsize τ, there exist $\delta, \hat{\delta}$ with bitsizes upper bounded by $\widetilde{O}(d^3(d+\tau))$, and R with bitsize upper bounded by $O(\mathrm{ht}(d) + \tau)$ such that Algorithm csos3 outputs an SOHS decomposition of f. The bitsize of the output coefficients is upper bounded by the output bitsize of the LDL^T decomposition of the matrix Q, that is $O(\hat{\delta}(d+1)^3) = \widetilde{O}(d^6(d+\tau))$. The running time is estimated as for csos2. □

6 PRACTICAL EXPERIMENTS

This section is dedicated to experimental results obtained by running our three certification algorithms, csos1, csos2 and csos3,

Algorithm 3 `csos3`

Input: $f \in \mathscr{H}(\mathbb{Z})[z]$ positive on $\mathscr{C}$ of degree d

Output: lists $[c_0, \ldots, c_d] \subset \mathbb{Q}_+$ and $[s_0, \ldots, s_d] \subset \mathbb{Q}[i][z]$ providing an SOHS decomposition of f as follows:

$$f = \sum_{k=0}^{d} c_k s_k^\star s_k$$

1: $\delta := 1, R := 1, \delta_c = 1, \hat{\delta} := 1$
2: boo := false
3: **while** not boo **do**
4: $(\tilde{Q}, \tilde{\lambda}) := \mathtt{sdp}(f, \delta, R)$
5: **if** $\tilde{\lambda} > 0$ **then** boo := true
6: **else** $\delta := 2\delta, R := 2R$
7: **end**
8: **done**
9: boo := false
10: **while** not boo **do**
11: $\hat{Q} := \mathtt{round}(\tilde{Q}, \hat{\delta})$
12: **for** $j \in \{0, \ldots, d\}, k \in \{0, \ldots, j\}$ **do**
13: $Q_{j,j-k} := \hat{Q}_{j,j-k} - \frac{1}{d-k+1}(\sum_{i=k}^{d} \hat{Q}_{i,i-k} - f_k)$
14: $Q_{j-k,j} := Q^\star_{j,j-k}$
15: **done**
16: $[c_0, \ldots, c_d; s_0, \ldots, s_d] := \mathtt{ldl}(Q)$ ▷ $f = \sum_{k=0}^{d} c_k s_k^\star s_k$
17: **if** $c_0, \ldots, c_d \in \mathbb{Q}_+, s_0, \ldots, s_d \in \mathbb{Q}[i][z]$ **then** boo := true
18: **else** $\hat{\delta} := 2\hat{\delta}$
19: **end**
20: **done**
21: **return** $[c_0, \ldots, c_d]$, $[s_0, \ldots, s_d]$

stated in Section 3, 4 and 5 respectively. First, we compare their performance to certify that trigonometric polynomials with Gaussian coefficients are positive on the unit circle $\mathscr{C}$. Next, we describe how to extend our third algorithm `csos3` to design a finite impulse response (FIR) filter in a certified fashion. Our code is implemented in Julia, freely available online [1], and the results are obtained on an Intel Xeon 6244 CPU (3.6GHz) with 1.5 TB of RAM. In `csos1` and `csos2`, we compute ε such that $f - \varepsilon$ is positive on $\mathscr{C}$ in Lines 2–3 by using msolve [4] within the Julia library `GroebnerBasis.jl`. The corresponding running time is denoted by t_ε. We denote by t_u the running time spent to compute the remainder polynomial u and to perform the comparison involving its coefficients and ε. In `csos1`, we compute approximate roots of $f - \varepsilon$ with the arbitrary-precision library `PolynomialRoots.jl` [34]. In `csos2` and `csos3`, we model SDP (12) via JuMP [10] and solve it with Mosek [1]. Exact arithmetic is performed with the Calcium library available in `Nemo.jl`.

6.1 Positivity verification

We consider a family of trigonometric polynomials having Gaussian integer coefficients $f_d = 10d + \sum_{k=1}^{d}((1-i)z^{-k} + (1+i)z^k)$. For each $d \in \{50, 100, 150, 200, 250\}$, we prove that f_d is positive on $\mathscr{C}$ by computing via `csos1`, `csos2` and `csos3` exact SOHS decompositions. Note that each such f_d is positive on $\mathscr{C}$ since $z^{-k} + z^k \geq -2$.

[1] https://homepages.laas.fr/vmagron/files/csos.zip

For `csos1`, we use a precision $\delta = 64$ (bits) to isolate complex roots. As a side note, we were not able to use arbitrary-precision SDP solvers (e.g., SDPA-GMP) within `csos2` and `csos3`, because JuMP only allows us to rely on double floating-point arithmetic at the moment. The running times (in seconds) of the 3 algorithms are reported in Table 1. As expected from the theoretical bit complexity results from Theorem 1 and Theorem 2, Algorithm `csos1` performs better than `csos2` and `csos3`. The reason why `csos2` is faster than `csos3` is due to the fact that the latter algorithm requires to perform an exact Cholesky's factorization. Even though `csos1` happens to be the best choice to verify the positivity of polynomials with known coefficients, the use of an SDP solver is mandatory to optimize over positive polynomials with unknown coefficients, as demonstrated in the next subsection.

	`csos1`			`csos2`			`csos3`
d	t_ε	t_u	total	t_ε	t_u	total	total
50	0.2	0.3	0.6	0.2	6.6	6.8	7.7
100	1.6	2.9	4.5	1.6	128	130	184
150	5.2	13	19	5.2	830	838	1460
200	24	26	51	24	3460	3485	7214
250	64	55	120	64	10553	10622	24852

Table 1: Performance of Algorithms `csos1`, `csos2`, and `csos3`

6.2 Design of a certified linear-phase FIR filter

This section is devoted to the design of a linear-phase finite impulse response (FIR) filter. This boils down to solving an energy minimization problem. To obtain a certified filter, we first solve a semidefinite optimization problem (corresponding to SDP (5.12) from [9]) and transform the numerical output into an exact certificate via a projection procedure similar to the one used in `csos3`.

Let $H(z) = \sum_{k=-d}^{d} h_k z^{-k}$ be an FIR filter of order d, with real coefficients. Let $h = [h_0, \ldots, h_d]$ be the coefficient vector of H. Since we work on the unit circle, we have $z = \exp(i\omega)$, for $\omega \in \mathbb{R}$, and we abuse notation by writing $H(\omega)$ instead of $H(z)$. The passband and stopband are respectively $[0, \omega_p]$ and $[\omega_s, \pi]$, where ω_p, ω_s are given. The stopband energy of the FIR filter is

$$E_s = \frac{1}{\pi}\int_{\omega_s}^{\pi} |h(\omega)|^2 d\omega.$$

To design such a linear-phase filter, we minimize the stopband energy under modulus constraints involving two parameters γ_p, γ_s:

$$\begin{aligned} \min_{H \in \mathscr{H}[z]} \ & E_s \\ \text{s.t.} \ & |H(\omega) - 1| \leq \gamma_p, \ \forall \omega \in [0, \omega_p], \\ & |H(\omega)| \leq \gamma_s, \qquad \forall \omega \in [\omega_s, \pi]\,. \end{aligned}$$

As shown in [9, § 5.1.1], the above problem can be reformulated as:

$$\begin{aligned} \min_{h, Q_1, \ldots, Q_7} \ & h^T \tilde{C} h \\ & (1+\gamma_p)1_{k=0} - h_k = L_k(Q1)\,, \\ & h_k - (1-\gamma_p)1_{k=0} = L_{k,0,\omega_p}(Q_2, Q_3)\,, \\ \text{s.t.} \ & \gamma_s 1_{k=0} - h_k = L_{k,\omega_s,\pi}(Q_4, Q_5)\,, \\ & \gamma_s 1_{k=0} + h_k = L_{k,0,\omega_p}(Q_6, Q_7)\,, \quad k = 0, \ldots, d\,, \\ & Q_1 \succeq 0, \ldots, Q_7 \succeq 0, \end{aligned} \tag{13}$$

where Q_1, Q_2, Q_4, Q_6 are real $(d+1)\times(d+1)$-matrices, Q_3, Q_5, Q_7 are real $(d-1)\times(d-1)$-matrices; $L_k(A) := \mathrm{tr}(\Theta_k A)$ and

$$L_{k,\alpha,\beta}(A,B) := \begin{array}{l}\mathrm{tr}(\Theta_k A) + \mathrm{tr}\Big(\big(\frac{a+b}{2}(\Phi_{k-1}+\Phi_{k+1})\\ -(ab+\frac{1}{2})\Phi_k - \frac{1}{4}(\Phi_{k-2}+\Phi_{k+2})\big)B\Big),\end{array} \quad (14)$$

with $a = \cos\alpha, b = \cos\beta$; $\Theta_k \in \mathbb{R}^{(d+1)\times(d+1)}, \Phi_k \in \mathbb{R}^{(d-1)\times(d-1)}$ are the elementary Toeplitz matrices with ones on the k-th diagonal and zeros elsewhere (they are zero matrices whenever k is out of range); $C = \mathrm{Toep}(c_0, \ldots, c_d)$ is the Toeplitz matrix with the first row $(c_0, \ldots, c_d)$, where

$$c_k = \begin{cases} 1 - \frac{\omega_s}{\pi}, & \text{if } k = 0,\\ -\frac{\sin k\omega_s}{k\pi}, & \text{if } k > 0,\end{cases}$$

$$\tilde{C} = P^T C P \succeq 0,\ P = \begin{bmatrix} 0 & J_d \\ 1 & 0 \\ 0 & I_d \end{bmatrix},$$

where J_d and I_d denote the counter identity and identity matrices of size d, respectively. By contrast with the unconstrained case (Algorithm `csos3`), this program involves 7 Gram real matrix variables and $d+1$ real variables $h_0, \ldots, h_d$, which are the coefficients of the polynomial corresponding to the filter.

After solving (13), we obtain numerical values for the coefficients of h and the entries of $Q_1, \ldots, Q_7$, which are further rounded to $\hat{h}$ and $\hat{Q}_1, \ldots, \hat{Q}_7$, respectively. To project $\hat{Q}_1$ to a matrix Q_1 satisfying exactly the first set of equality constraints in SDP (13), we apply the formula in Line 13 of Algorithm `csos3` after replacing f_k by $p_k := (1+\gamma_p)1_{k=0} - \hat{h}_k$. Similarly, we obtain matrices Q_2 and $Q_3 := \hat{Q}_3$ satisfying exactly the second set of equality constraints in SDP (13), after substitution by

$$\begin{array}{l}\hat{h}_k - (1-\gamma_p)1_{k=0} - \mathrm{tr}\Big(\big(\frac{a+b}{2}(\Phi_{k-1}+\Phi_{k+1}) - (ab+\frac{1}{2})\Phi_k\\ -\frac{1}{4}(\Phi_{k-2}+\Phi_{k+2})\big)Q_3\Big),\end{array}$$

Eventually, similar projection steps provide the remaining matrices $Q_4, \ldots, Q_7$ so that all equality constraints in (13) hold exactly.

As in [9, Example 5.1], we design a filter with parameters $d = 25$, $\omega_p = \pi/5$, $\omega_s = \pi/4$, $\gamma_p = 1/10$ (which corresponds to a passband ripple of 1.74 dB), and $\gamma_s = 0.0158$ (a stopband attenuation of 36 dB). We first obtain a numerical lower bound of the stopband energy $E_s' = 4.461501 \times 10^{-5}$. However, this bound happens to be inexact as the Gram matrices obtained after the projection step are not positive semidefinite anymore. To overcome this certification issue, we replace the last constraint in (13) by $Q_7 - 10^{-9} I_{24} \succeq 0$. Doing so, we can successfully project the approximate Gram matrices into exact ones with positive eigenvalues, and obtain a certified exact lower bound of $E_s = 4.461503 \times 10^{-5}$ in 0.74 seconds.

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