Exact algorithms for structured polynomial optimization

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More info and positions at https://easychair.org/cfp/POEMA-19-22.
Contact at Sorbonne University: Mohab Safey El Din `Mohab.Safey@lip6.fr`

Research Context. Optimizing an algebraic function over algebraic constraints is a topical issue in at the interplay of mathematics and computational aspects of computer science. It arises in a wide span of applications of engineering sciences for which getting the right optimum is crucial. In this context, exactness is a challenge due to the intrinsic non-linearity of the problems.

During the last decade, several promising techniques have emerged in this new field, leading to striking developments from a cross fertilization between (real) algebraic geometry, applied mathematics, theoretical computer science and engineering.

The use of algebraic computational techniques for solving polynomial systems of equations have enabled the design of exact algorithms enabling to compute exactly the infimum of a polynomial function over sets in $\mathbb{R}^n$ defined by polynomial equations or certify that such an infimum does not exist. The practical behaviour of these algorithms reflect their asymptotically optimal complexity.

The challenging goal of this internship and its subsequent PhD is to design algorithms to tackle first the important case of semi-algebraic constraints from the application point of view and significantly improve the state-of-the-art algorithms for taking advantage of typical algebraic structures arising in applications (multi or weighted homogeneity, symmetries, compositions, etc.).

Goals. Preliminary work will consist of studying the exisiting algorithms to solve exactly polynomial optimization problems. In particular, they rely on the so-called critical point method whose most recent variants rely on Gröbner bases computations combined with advanced methods in commutative algebra, algebraic geometry and differential geometry [2, 3].

A promising fruitful research direction for generalizing these algorithms to the semi-algebraic set case is then to adapt geometric and combinatorial approaches of [1, 4] for deciding the consistency over the reals to systems of polynomial inequalities to the problem of polynomial optimization.

In a next step, one will relate these algorithmic strategies with the notions of generalized critical values introduced by Kurdyka, Orro and Simon [5] to obtain variants of Ehresmann's fibration theorem to non-proper situations. In particular, this will allow to design algorithms deciding the existence of an infimum. A preliminary implementation based on routines provided in `RAGlib` [6]. is expected and will be used to solve some challenging applications.

Further research will lead the candidate to use similar connections in order to obtain better algorithms taking advantage of algebraic structures such as weighted or multi-homogeneity of the target function and/or the constraints, symmetries. A particular focus on polynomial optimization problems whose algebraic constraints are obtained by composition will also be considered. This will lead the candidate to deeply study the behaviour of algebraic algorithms for solving polynomial systems such as Gröbner bases algorithms.

Working Context. The PhD candidate will be hosted by the PolSys team, which is a joint team of CNRS (LIP6), Inria and Sorbonne Université. It is located at Campus Jussieu, in the heart of Paris (5-th district). The group, led by Jean-Charles Faugère, is internationally recognized for major contributions in the area of solving systems of polynomial equations/inequalities using exact methods. It is used to welcome international students in a nice and enjoyable working atmosphere.

Planned secondments. The PhD candidate will have a research stay (secondments) at Univ. of Konstanz (M. Schweighofer) and NAG (M. Dewar).

Required Skills. Motivated candidates should hold a Bachelor degree and have a solid background in **either** optimization, real algebraic geometry or computer algebra. Good programming skills are also a plus. The candidates are kindly asked to send an e-mail with "POEMA candidate" in the title, a CV and motivation letter to `mohab.safey@lip6.fr`. Knowledge of French does not constitute a pre-requisite at

all.
Official submission link: https://easychair.org/conferences/?conf=poema1922

References

[1] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in real algebraic geometry*, volume 10. Springer Science & Business Media, 2007.

[2] A. Greuet, F. Guo, M. Safey El Din, and L. Zhi. Global optimization of polynomials restricted to a smooth variety using sums of squares. *Journal of Symbolic Computation*, 47(5):503 – 518, 2012.

[3] A. Greuet and M. Safey El Din. Probabilistic Algorithm for Polynomial Optimization over a Real Algebraic Set. *SIAM Journal on Optimization*, 24(3):1313–1343, 2014.

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[5] K. Kurdyka, P. Orro, S. Simon, et al. Semialgebraic sard theorem for generalized critical values. *Journal of differential geometry*, 56(1):67–92, 2000.

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