Algebraic tools for exact SDP and its variants

This PhD is funded by the Marie Curie program of European Union through the innovative training network (ITN) POEMA on polynomial optimization.
More info and positions at https://easychair.org/cfp/POEMA-19-22.
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Research Context. Certification and validation of computational results is a major issue for modern sciences raising challenging problems at the interplay of mathematics and computational aspects of computer science. One can emphasize in this context several applications arising in the design of modern cyber-physical systems with a crucial need of *exact* certification. These issues give rise to many mathematical problems. Polynomial optimization (which consists in computing the infimum of a polynomial function under algebraic constraints) is one of the most important, difficult and challenging one.The emergence of this exciting new field goes back to the last decade and has led to striking developments from a cross fertilization between (real) algebraic geometry, applied mathematics, theoretical computer science and engineering.

Consider for instance the problem of minimizing $4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$ over $\mathbb{R}^2$. One way to certify that its minimum is 0 is to decompose this polynomial as a *sum of squares* (SOS), which is the core subject of study in real algebra. Here the decomposition is $(2xy + y^2)^2 + (2x^2 + xy - 3y^2)^2 \geq 0$. In general, one can compute such SOS decompositions by solving a *semi-definite program* (SDP) [2], which is a standard tool in applied mathematics and convex optimization. In SDP, one optimizes a linear function under the constraint that a given matrix is semi-definite positive, i.e. has only non-negative eigenvalues. One particular issue arising while relying on SDP solvers is that they are numerical approximate routines, thus output only *approximations* of the certificates.

The challenging goal of this internship is to design algorithms to compute *exact* certificates while controlling the bit complexity of the algorithmic procedures.

Goals. Preliminary work will consist of studying the exisiting algorithms to obtain exact SOS decompositions of non-negative polynomials. In particular, the case of univariate polynomials has been recently handled in [5] by means of classical techniques from symbolic computation (real root isolation, square-free decomposition). An extension to multivariate polynomials has been derived in [3] thanks to a perturbation/compensation algorithm. A promising research track would be to apply the certification algorithms from [5] to a multivariate polynomial through a reduction to the univariate case. That reduction exploits algebraic properties of multivariate polynomial systems and Gröbner bases algorithms.

The idea is to characterize the set of minimizers of this polynomial by exploiting the information given by the Jacobian, in the same spirit as in [6]. After designing the certification framework, further efforts should lead to provide the related bit complexity estimates, both on runtime and output size. Practical experiments shall be performed through implementing a tool within the Maple libraries `RealCertify` [4] and `RAGlib` [7].

Further research will lead the candidate to use similar connections between semi-definite programming and algebra, both for enhancing certification algorithms and analyzing their behaviour. We will also focus on algebraic structures arising in applications (such as multi-symmetry and multi-homogeneity). This will lead the candidate to consider algebraic properties of intrinsic objects such as the central curve related to semi-definite programming (see e.g. [1]) and the use of homotopy techniques for solving LMIs.

Official submission link: `https://easychair.org/conferences/?conf=poema1922`

Working Context. The PhD candidate will be hosted by the PolSys team, which is a joint team of CNRS (LIP6), Inria and Sorbonne Université. It is located at Campus Jussieu, in the heart of Paris (5-th district). The group, led by Jean-Charles Faugère, is internationally recognized for major contributions in the area of solving systems of polynomial systems using exact methods. It is used to welcome international students in a nice and enjoyable working atmosphere.

Planned secondments. The PhD candidate will have a research stay (secondments) at Univ. of Firenzi (G. Ottaviani) and RTE (J. Maeght).

Required Skills. Motivated candidates should hold a Bachelor degree and have a solid background in **either** optimization, real algebraic geometry or computer algebra. Good programming skills are also a plus. The candidates are kindly asked to send an e-mail with "POEMA candidate" in the title, a CV and motivation letter to `mohab.safey@lip6.fr`. Knowledge of French does not constitute a pre-requisite at all.

References

[1] D. Henrion, S. Naldi, and M. Safey El Din. Exact Algorithms for Linear Matrix Inequalities. *SIAM Journal on Optimization*, 26(4):2512–2539, 2016.

[2] J.-B. Lasserre. Global Optimization with Polynomials and the Problem of Moments. *SIAM Journal on Optimization*, 11(3):796–817, 2001.

[3] V. Magron and M. Safey El Din. On Exact Polya and Putinar's Representations. In *ISSAC'18: Proceedings of the 2018 ACM International Symposium on Symbolic and Algebraic Computation*. ACM, New York, NY, USA, 2018.

[4] V. Magron and M. Safey El Din. RealCertify: a Maple package for certifying non-negativity. In *ISSAC'18: Proceedings of the 2018 ACM International Symposium on Symbolic and Algebraic Computation*, ***Best Software Demo Award***. ACM, New York, NY, USA, 2018.

[5] V. Magron, M. Safey El Din, and M. Schweighofer. Algorithms for weighted sum of squares decomposition of non-negative univariate polynomials. *Journal of Symbolic Computation*, 2018.

[6] J. Nie. An exact Jacobian SDP relaxation for polynomial optimization. *Mathematical Programming*, 137(1):225–255, 2013.

[7] RAGlib – A library for real solving polynomial systems of equations and inequalities. `http://www-polsys.lip6.fr/~safey/RAGLib/distrib.html`.