Master 2 internship Project
Computational real algebraic geometry
with a view towards optical system design

We are seeking an excellent candidate for a 5 to 6-month research internship at the interface of computational mathematics, algebraic and real algebraic geometry, with the goal of solving challenging problems which arise in optical system design. This internship is proposed to **students willing to pursue a Ph.D.** after obtaining their Master degree, through a collaboration between the PolSys team (Polynomial Systems) of Sorbonne Université and CNRS with the company Thalès Aliena space.

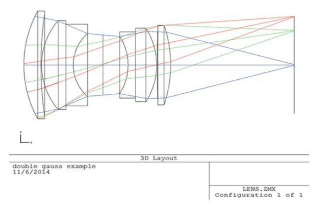
Scientific context. Optical system design is a key component of advanced technologies which are foundational for everyday applications spanning from cameras and telescopes to medical equipment, automative sensors and satellites, space applications being of major interest to Thalès Aliena space. This boils down to combine lenses, mirrors and other optical components of the system with the goal to optimize some optical and geometric criteria such as aberration, i.e. deviations from ideal image formation due to imperfections in the system, or feasibility criteria of both technical and physical aspects realization.

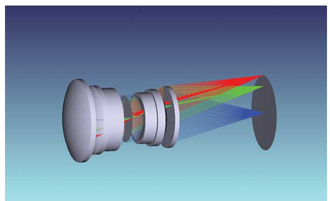
Figure 1: Optical inter-satellite links by Thalès Aliena Space

A crucial task in optical system design is the **classification of architectures of optical systems** according to optical features (number of mirrors, their relative position and their nature – spherical or not, etc.), which is a key assett to the engineer for building optimal optical systems. A first issue comes from the **modeling of such classification problems** which is to be done with care, as physical and geometric phenomena are in the middle of the scene.

However, a remarkable fact is that most of optical laws are governed by Euclidean geometry. Hence, turning **classification of optical systems into equations and constraints lead to systems of polynomial constraints** with real coefficients. The set of **real solutions** to such polynomial systems is called the set of admissible solutions. All points in a given such connected component correspond to an optical system. All points in the same connected component correspond to optical systems which share similar optical features.

This gives rise to a **strong connection between optical system design and computational real algebraic geometry**. Indeed, computational aspects are ubiquitous in real algebraic geometry which is the area of mathematics studying the class of semi-algebraic sets (which are sets defined by finitely many polynomial constraints with real coefficients) and maps between semi-algebraic sets. A witness of this is the following result: the connected components of real solutions to systems of polynomial equations and inequalities with real coefficients are finitely many [2, Theorem 2.4.4].





Hence, it does make sense to enumerate them by computing at least one point in each of these connected components, given a defining system of polynomial equations and inequalities. Since these connected components are in one-to-one correspondence with optical features, there is also a need to answer connectivity queries over those connected components (i.e. given two points in the semi-algebraic set under study, decide whether they can be path connected in a component of that semi-algebraic set). Again, this is an algorithmic task.

Ultimately, since connected components of semi-algebraic sets are also semi-algebraic sets [2, Theorems 2.4.4 and 2.4.5], one can expect to compute polynomial systems defining these connected components individually.

These both algorithmic tasks allow us to obtain a **nomenclature** of optical systems.

Problem, state of the art, objectives. This internship targets the first steps towards the design and the implementation of complete algorithms for solving classification problems in optical system design.

The current state of the art is as follows. Currently, only the case of $N = 3$ mirrors is solved in the litterature, under the assumption that all three mirrors lie on one axis and obscuration phenomena are neglicted [5]. The current modeling of this case leads to study a semi-algebraic set lying in $\mathbb{R}^5$. The study of its connected components is obtained with ad-hoc methods which are very much specific to the shape of the polynomial system. It is more than unlikely that these methods could be used for more general and challenging settings.

To go further, there is a need of **algorithms** that can be used on a wide range of polynomial systems, for solving the aforementioned problems:

- SamplePoints: this algorithm takes as input a system of polynomial equations and inequalities with real coefficients and returns an encoding of at least one point per connected component in the semi-algebraic set defined by the input polynomial constraints.

  For instance, on input the polynomial inequality $x_1x_2 - 1 > 0$, such an algorithm could return the set of points $\{(1, 2), (2, 1), (-1, -2), (-2, -1)\}$.

- Connect: this algorithm takes as input a system of polynomial equations and inequalities with real coefficients defining a semi-algebraic set S with a finite set of points $\mathscr{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_\ell\}$ with $\boldsymbol{x}_i \in S$ and returns a partition $\mathscr{P}_1, \ldots, \mathscr{P}_k$ of $\mathscr{P}$ such that $(\boldsymbol{x}_i, \boldsymbol{x}_j) \in \mathscr{P}_r \times \mathscr{P}_r$ if and only if there exists a connected component C of S such that $\boldsymbol{x}_i \in C$ and $\boldsymbol{x}_j \in C$.

  In other words, Connect gathers the points of $\mathscr{P}$ which lie in the same connected component of S.

  Giving as input to Connect, the polynomial inequality $x_1x_2 - 1 > 0$ and the finite set of points $\mathscr{P} = \{(1, 2), (2, 1), (-1, -2), (-2, -1)\}$ returned by SamplePoints, Connect returns $\mathscr{P}_1 = \{(1, 2), (2, 1)\}$ and $\mathscr{P}_2 = \{(-1, -2), (-2, -1)\}$.

  Note that, combined with SamplePoints, the algorithm Connect provides a way to obtain a nomenclature of optical systems which share a similar architecture.

- Describe: this algorithm takes as input a system of polynomial equations and inequalities with real coefficients, defining a semi-algebraic set S, and returns a family of polynomial constraints $\mathscr{C}_1, \ldots, \mathscr{C}_k$ such that their respective real solution sets $S_1, \ldots, S_k$ are the connected components of S.

  For instance, on input $x_1x_2 - 1 = 0$, one would expect as an output $\mathscr{C}_1 = \{x_1x_2 - 1 = 0, x_1 > 0\}$ and $\mathscr{C}_2 = \{x_1x_2 - 1 = 0, x_1 < 0\}$.

Such algorithms already exist and can be found in [1]. They are based on **computer algebra paradigms** and make use of **Gröbner bases** computations to put into practice algebraic elimination theory in order to **reduce the initial problem** stated in arbitrary dimension **to equivalent problems but with small dimension**.

However, all these algorithms suffer from the curse of dimensionality. Letting n be the dimension of the ambient space and d be the maximum degree of the input polynomials and s be the cardinality of the polynomial constraints, the following holds:

- SamplePoints runs in time $(sd)^{O(n)}$ arithmetic operations;
- Connect runs in time $(sd)^{O(n^2)}$ arithmetic operations;
- Describe runs in time $(sd)^{O(n^3)}$ arithmetic operations.

There also exist quantitative estimates which show that it cannot be hoped much better than complexities which are singly exponential in n since the number of connected components of semi-algebraic sets defined by polynomials in degree d and n variables can be exponential in n and polynomial in d.

Besides, there is no existing implementation of the algorithms described in [1] that can solve non trivial examples because the Landau notation used here in the exponent hides rather large constants.

This is why innovative algorithms have been designed since the last twenty years to **obtain efficient implementations** that scale enough to solve non-trivial problems coming from engineering applications. This is the case for the SamplePoints problem [11] in the case of real algebraic sets and for the Connect problem [9,10] in the case of real algebraic sets too. The problem Describe has been much less studied but is related to recent progress on algorithms classifying the number of real roots of systems depending on parameters [6,8].

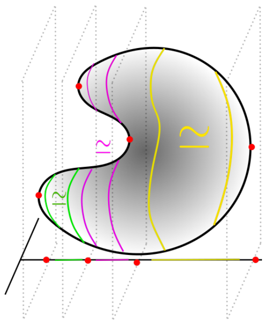


Figure 2: Illustration of topological invariance property

These new algorithms and their implementations have already allowed us to solve challenging problems coming from engineering sciences such as biology [7,12] and robotics [3,4].

The practical efficiency of their implementations comes from several ingredients. First, the topology of the sets under study is precisely analysed using advanced results from **Morse theory** and **differential geometry** which leverage properties of **critical points** of some well-chosen polynomial maps. This is illustrated by Figure 2, which shows that all fibers of the projection on the first coordinate axis, taken between critical values of that projection, are diffeomorphic to each other.

These **critical points** are then encoded with systems of polynomial equations which are solved using dedicated **Gröbner bases** techniques. This is where **algebraic geometry** and **commutative algebra** play a crucial role.

The **goal of this internship** is to push forward, in several ways, the current capabilities of algorithms for solving polynomial systems of equations and inequalities over the reals in order to solve the problems which arise in optical system design.

To do so, one will start by studying carefully how the current algorithms and behave on the case $N = 3$ mirrors, all on the same axis, which is already solved in the litterature [5]. The expectation is to retrieve here automatically the results obtained with ad-hoc methods. This might bring a novel understanding on the geometry of the problem and alternative formulations for the achieved nomenclature of optical systems in that case.

Moreover, this step will pave the way to understand the current limitations of existing algorithms to obtain the nomenclature of architectures of optical systems involving $N = 4$ mirrors. Hence, a next step will be to study the structure of the polynomial systems we have at hand to start to obtain a nomenclature of optical systems in that case. What we expect is a first version of the SamplePoints routine generalizing and adapting the one of [11] to the case of systems of polynomial equations *and* inequalities. Next, we will study variants of algorithms for the problem Connect and Describe using algorithmic ingredients introduced in [8,9] and [6]. We target first implementations of these algorithms at least in some simplified cases and the treatment of polynomial systems coming from optical system design involving $N = 4$ mirrors, which will go beyond the state of the art.

Software implementations. The computer algebra community has produced several high-quality software libraries which provide high-performance basic tools. They are widely used, most often because they are called as subroutines in software for another more specific task, or are integrated in general computer algebra systems such as macaulay2, singular, `msolve` for multivariate polynomials and commutative algebra. Several of these have received the ACM/SIGSAM Richard D. Jenks Memorial Prize. In this project, we will make use of computer algebra systems such as SageMath or Maple or the AlgebraicSolving.jl Julia package for prototyping and experimentation linked with the C fast libraries such as `msolve` and `Flint`.

Scientific environment. This internship will be co-supervised by Mohab Safey El Din (Sorbonne Université) and Audric Drogoul (Thalès Aliéna Space). It will take place at Sorbonne Université, in the computer science lab LIP6, which is located on the Pierre-et-Marie-Curie campus, at the heart of Paris. The intern will be welcome in the PolSys team, which develops and implements fast computer algebra algorithms, for polynomial system solving and their applications, such as coding theory, combinatorics, cryptography, and robotics. The intern will work in a kind and international environment gathering PhD students and Post-Docs representing several nationalities, and animated with several working groups and a monthly seminar. All computing facilities will be provided.

Weekly meetings with the supervision team will be organized and some trips to the R&D offices in Cannes (South of France) will be organized too.

Prerequisites. We are looking for exceptional candidates who are excited about computational algebraic and real algebraic geometry, in a broad sense, and its applications to geometric problems as those arising in optical system design. We expect curiosity and openness, experience with mathematical software, strong interpersonal and communicative skills, and willingness to dive into new topics, including the implementation of powerful mathematical algorithms.

No prerequisite in physics or optics is required. The applicants should have a solid backround in mathematics with some specialization in either geometry (in a broad sense, differential, algebraic or real algebraic geometry) or algebra or optimization. No strong programming skills are required but an experience with mathematical software is appreciated and willingness to develop implementations of mathematical algorithms is mandatory.

How to apply. Applicants should send a full CV, a letter of motivation and the grades obtained during the last two years to

mohab.safey@lip6.fr, audric.drogoul@thalesaleniaspace.com

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