Heights of degeneracy loci

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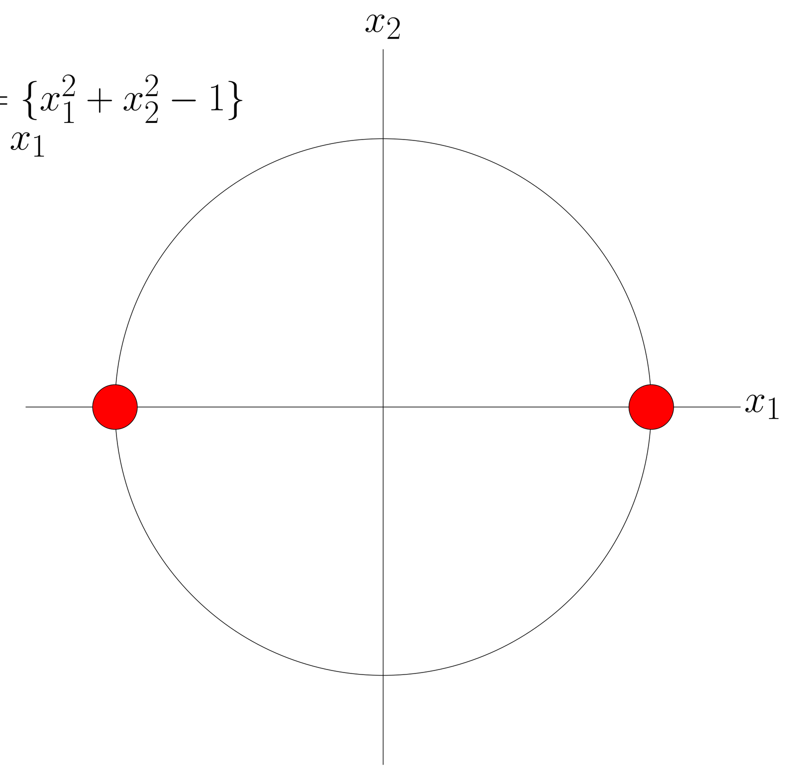
Critical Points and Gröbner Bases

Problem

Compute the critical points of a polynomial g restricted to an algebraic set $V(F)$.

Critical points of $g_{|V(F)}$ ⟷ Variety defined by F and the **maximal minors** of $\mathrm{jac}(g, F)$.

$F = \{x_1^2 + x_2^2 - 1\}$
$g = x_1$



$$\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \end{pmatrix}$$
$$\Downarrow$$
$$\begin{cases} f_{11}f_{22} - f_{12}f_{21} \\ f_{11}f_{23} - f_{13}f_{21} \\ f_{11}f_{24} - f_{14}f_{21} \\ f_{12}f_{23} - f_{13}f_{22} \\ f_{12}f_{24} - f_{14}f_{22} \\ f_{13}f_{24} - f_{14}f_{23} \end{cases}$$

(Arithmetic) intersection theory

Degree [Fulton, 1998]

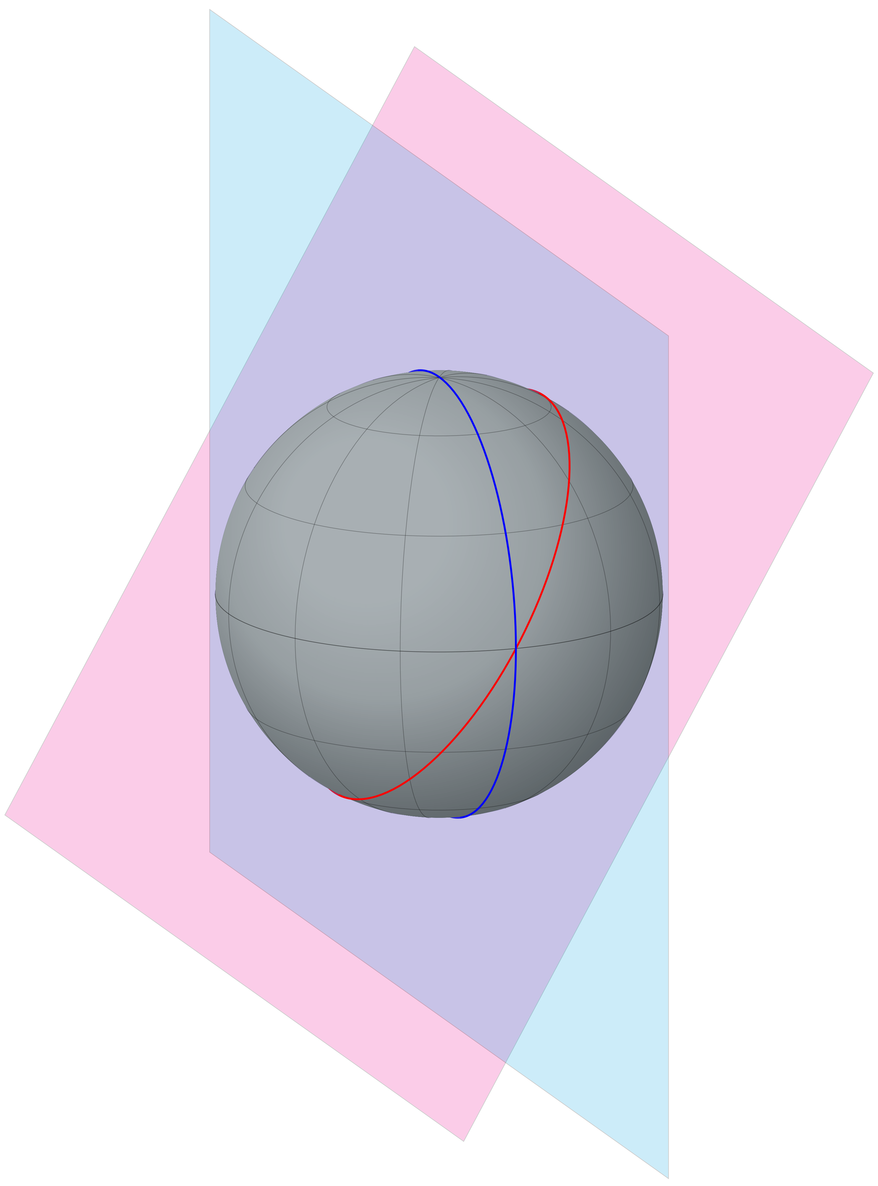
A variety V of dimension d has $\deg(V)$ points of intersection with d general hyperplanes.

$\mathbb{Z}$

Height [Bost-Gillet-Soulé, 1994]

Each such point of intersection has bit size bounded from above by the *height* $h(V)$.

$\mathbb{R}$



Example

Two general planes intersect a sphere in two points.

Setup

- $S = \operatorname{Spec} \mathbb{Z}$.
- Integers $1 \leq e \leq f$.
- Degrees $\delta_1, \ldots, \delta_f$.
- $N = f - e$.
- $E = \mathcal{O}_S^{\oplus N+1}$, so $\mathbb{P}(E) \cong \mathbb{P}_{\mathbb{Z}}^N$.
- $\mathcal{E} = \mathcal{O}_{\mathbb{P}(E)}^{\oplus e}$.
- $\mathcal{F} = \bigoplus_{i=1}^f \mathcal{O}_{\mathbb{P}(E)}(\delta_i)$.

An incidence correspondence

Problem

How do we encode coefficients which define matrices with nonempty first degeneracy loci using the incidence correspondence of points on hyperplanes?

Suppose we have

$$[X_0 : X_1] \in \mathbb{P}^1$$
$$[a_{0,0} : b_{0,0} : a_{0,1} : b_{0,1}], [a_{2,0} : b_{2,0} : a_{2,1} : b_{2,1}] \in \check{\mathbb{P}}^3$$
$$[a_{1,0} : b_{1,0} : c_{1,0} : a_{1,1} : b_{1,1} : c_{1,1}] \in \check{\mathbb{P}}^5$$
$$[\lambda_0 : \lambda_1] \in \mathbb{P}^1$$

such that

$$\begin{bmatrix} a_{0,0}X_0 + b_{0,0}X_1 & a_{0,1}X_0 + b_{0,1}X_1 \\ a_{1,0}X_0^2 + b_{1,0}X_0X_1 + c_{1,0}X_1^2 & a_{1,1}X_0^2 + b_{1,1}X_0X_1 + c_{1,1}X_1^2 \\ a_{2,0}X_0 + b_{2,0}X_1 & a_{2,1}X_0 + b_{2,1}X_1 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \end{bmatrix} = 0$$

Then, for example:

$$a_{1,0}\left(\lambda_0 X_0^2\right) + b_{1,0}(\lambda_0 X_0 X_1) + c_{1,0}\left(\lambda_0 X_1^2\right) + a_{1,1}\left(\lambda_1 X_0^2\right) + b_{1,1}(\lambda_1 X_0 X_1) + c_{1,1}\left(\lambda_1 X_1^2\right) = 0.$$

Solution

We consider the incidence correspondence of points on hyperplanes restricted to the Segre embedding of $\mathbb{P}^N \times_S \mathbb{P}^{e-1} \hookrightarrow \mathbb{P}^{e(N+1)}$.

Example setup

- $e = 2, f = 3$
- $\delta_1 = 1, \delta_2 = 2, \delta_3 = 1$

A determinantal A-resultant

Product of incidence correspondences of points on hyperplanes. ⇢ Γ

$$\Gamma \xrightarrow{p} \prod_{i=1}^f \mathbb{P}_i, \qquad \Gamma \xrightarrow{q} \prod_{i=1}^f \check{\mathbb{P}}_i$$

Product of Segre embeddings.

$$\prod_{i=1}^f \left(\mathbb{P}(\mathcal{O}_S^{\oplus e}) \times_S \mathbb{P}\left(\operatorname{Sym}^{\delta_i} E^\vee\right)^\vee\right) \xrightarrow{s = \prod_{i=1}^f s_i} \prod_{i=1}^f \mathbb{P}_i$$

Product of Veronese embeddings.

$$\mathbb{P}(\mathcal{O}_S^{\oplus e}) \times_S \mathbb{P}(E) \xrightarrow{\nu = \prod_{i=1}^f (\mathrm{id} \times \nu_i)} \prod_{i=1}^f \left(\mathbb{P}(\mathcal{O}_S^{\oplus e}) \times_S \mathbb{P}\left(\operatorname{Sym}^{\delta_i} E^\vee\right)^\vee\right)$$

$$\operatorname{Res}_{\mathcal{E},\mathcal{F},e-1} = q_* p^* s_* \nu_* \left(\mathbb{P}(\mathcal{O}_S^{\oplus e}) \times_S \mathbb{P}(E)\right) \dashrightarrow \prod_{i=1}^f \check{\mathbb{P}}_i$$

Notation

$$\mathbb{P}_i = \mathbb{P}\left(\mathcal{O}_S^{\oplus e} \otimes \left(\operatorname{Sym}^{\delta_i} E^\vee\right)^\vee\right)$$

Fact [Busé, 2002]

When $N = f - e$, the determinantal resultant has codimension one in $\prod_{i=1}^f \check{\mathbb{P}}_i$.

Main theorem

Notation

- $\eta_k(x_1, \ldots, x_f)$ is the k^{th} elementary symmetric polynomial.
- $\eta_k^{(i)}(x_1, \ldots, x_f) = \eta_k(x_1, \ldots, \widehat{x_i}, \ldots, x_f)$.
- $\sigma_p = \frac{(p+1)}{2} \sum_{m=1}^p \frac{1}{m} - \frac{p}{2}$ is the p^{th} *Stoll number*.

The **multidegree** of the determinantal resultant is

$$\deg_i\left(\operatorname{Res}_{\mathcal{E},\mathcal{F},e-1}\right) = \eta_{f-e}^{(i)}(\delta_1, \ldots, \delta_f).$$

The **normalized multiheight** (see [Bost-Gillet-Soulé, 1994]) of the determinantal resultant is

$$h\left(\operatorname{Res}_{\mathcal{E},\mathcal{F},e-1}\right) = \eta_{e-1}(\delta_1, \ldots, \delta_f)\sigma_{f-e} + \eta_{f-e}(\delta_1, \ldots, \delta_f)\sigma_{e-1}.$$

The **logarithmic Mahler measure** of the determinantal resultant is

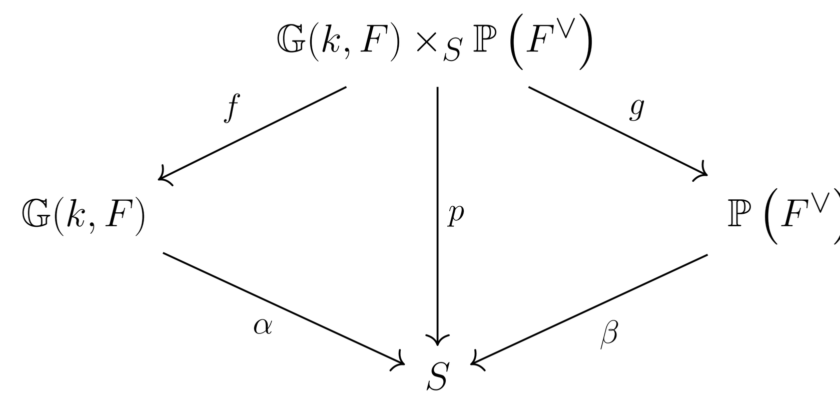
$$\int_{\left(\prod_{i=1}^f \check{\mathbb{P}}_i\right)(\mathbb{C})} \log\left|\operatorname{Res}_{\mathcal{E},\mathcal{F},e-1}\right| d\prod_{i=1}^f \mu_i^{e\binom{N+\delta_i}{\delta_i}-1} = h\left(\operatorname{Res}_{\mathcal{E},\mathcal{F},e-1}\right) - \frac{1}{2}\sum_{i=1}^f \left(\deg_i\left(\operatorname{Res}_{\mathcal{E},\mathcal{F},e-1}\right) \sum_{j=1}^{e\binom{N+\delta_i}{\delta_i}-1} \frac{1}{j}\right)$$

Higher degeneracy loci

Fact

An $f \times e$ matrix M has rank at most r if and only if $\dim(\ker(M)) \geq e - r$.

→ Replace $\mathbb{P}\left(\mathcal{O}_S^{\oplus e}\right)$ with $\mathbb{G}\left(e - r, \mathcal{O}_S^{\oplus e}\right)$.



Theorem

Let I be the incidence correspondence of k-planes contained in hyperplanes in $\mathbb{G}(k, F) \times_S \mathbb{P}\left(F^\vee\right)$. Then $I^*(c(Q)) = c(S^\vee)$.

Heights
↓
Arithmetic Schubert calculus
[Maillot, 1995], [Tamvakis, 1999]

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